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Prime Jordan Algebras Satisfying Local Goldie Conditions*

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1. INTRODUCTION

Based on ideas from semigroup theory, Fountain and Gould [11, 12] introduced a notion of order in a ring which need not have a unit and gave (see [12] or [13]) a Goldie-like characterization of two-sided orders in semisimple rings with minimum condition on principal one-sided ideals. Later, Anh and Marki [2] extended these results to the one-sided case.

Inspired by these works, we introduce in the present paper a notion of local order in a Jordan algebra which need not have a unit and prove that a Jordan algebra is a local order in a simple Jordan algebra with descending chain condition (dcc) on principal inner ideals but which is not a nonartinian quadratic factor if and only if it is a prime nondegenerate Jordan algebra satisfying local Goldie conditions. Actually, our result is a natural extension of a celebrated Goldie theorem for Jordan algebras due to Zel'manov [30, 31].

The paper is organized as follows: Section 2 is devoted to local orders in (associative) rings. For the sake of completeness, we give in this section the necessary definitions and state those theorems that will be used later. Section 3 deals with orders in Jordan algebras in Zel'manov's sense. Again this section just provides those definitions and basic results that will be needed in the remainder of the paper. In Sections 4 and 5 we introduce the notions of weak local order and local order in Jordan algebras (not necessarily with a unit) and prove (Theorem 20) that if a Jordan algebra J is a weak local order in a nondegenerate (respectively simple) Jordan algebra Q with dcc on principal inner ideals, then J is a nondegenerate (respectively prime nondegenerate) Jordan algebra satisfying ascending

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chain condition (acc) on the annihilators of its elements. Moreover, if Q does not contain nonartinian quadratic factors then every element x in J has finite Goldie dimension, i.e., $U_x J$ contains no infinite direct sum of inner ideals. The aim of Section 6 is to prove the converse of Theorem 20 for the case of a prime nondegenerate Jordan algebra. The key tools we use in order to obtain this result are Zel'manov's classification of prime nondegenerate Jordan algebras and those results on local orders in associative algebras given in Section 2. In the case that J is trapped between the hermitian elements $H(A, *)$ of a prime associative algebra A relative to a diagonal involution $*$ and those of the symmetric ring of quotients $Q_s(A)$ of A, we can use Zel'manov's methods to prove that A is a local order in a simple associative algebra Q with minimal one-sided ideals, and hence that J is a local order in $H(Q, *)$. However, if the involution $*\colon A \to A$ is alternate, these methods do not work, but then A satisfies a generalized identity with involution and hence $Q_s(A)$ has nonzero socle B. By using results of Section 2, we prove that in this case J is a local order in the simple Jordan algebra $H(B, *)$.

2. Local Orders in Associative Rings

Throughout this section *ring* will mean associative ring (not necessarily with a unit). Let R be a ring. Recall that $a \in R$ is *regular* if it is not a zero divisor,

$$ax = 0 \Rightarrow x = 0 \qquad \text{and} \qquad xa = 0 \Rightarrow x = 0$$

for $x \in R$. We denote by $\mathrm{Reg}(R)$ the set of regular elements of R and by $\mathrm{Inv}(Q)$ the invertible elements of a unital ring Q.

A subring R of Q is a *classical order* in Q if

$$\mathrm{Reg}(R) \subseteq \mathrm{Inv}(Q), \text{ and} \tag{2.1}$$

$$\text{for every } q \in Q, \qquad q = a^{-1}b = cd^{-1}, \qquad \text{for } b, c \in R,\ a, d \in \mathrm{Reg}(R). \tag{2.2}$$

Note that we can always find a common denominator, i.e., (2.2) is equivalent to

$$\text{for every } q \in Q, \qquad q = s^{-1}r = ts^{-1}, \qquad \text{for } r, t \in R,\ s \in \mathrm{Reg}(R) \tag{2.3}$$

(take $s = da$, $r = db$, $t = ca$ from (2.2)).

An element $a \in R$ is called *semiregular* if

$$a^2x = 0 \Rightarrow ax = 0, \quad \text{and} \quad xa^2 = 0 \Rightarrow xa = 0,$$

for $x \in R'$ (the unitization of R). We denote by SemiReg(R) the set of all semiregular elements of R. Certainly SemiReg(R) $\supseteq$ Reg(R). We remark that if $a \in a^2Q \cap Qa^2$ for some over-ring $Q \supseteq R$ then $a \in$ SemiReg(R).

Let LocInv(R) denote the set of elements $a \in R$ which are *locally invertible* in the sense that there exists an idempotent $e \in R$ such that a is invertible in the unital ring eRe. Then the *local inverse* $a^{\#} \in eRe$ is precisely the *group inverse* of a, and it is characterized by the conditions

$$aa^{\#} = a^{\#}a, \qquad a = aa^{\#}a, \qquad a^{\#} = a^{\#}aa^{\#}.$$

The idempotent e is also unique, $e = aa^{\#} = a^{\#}a$. Moreover, a is locally invertible if and only if $a \in a^2Ra^2$ (see [10]). Thus, by above remark, if $a \in R$ is locally invertible in some over-ring Q, then a is semiregular in R:

$$R \cap \mathrm{LocInv}(Q) \subseteq \mathrm{SemiReg}(R).$$

Conversely,

PROPOSITION 1 [11, Proposition 2.6]. *Let R be a ring satisfying dcc on principal right ideals. Then every semiregular element $x \in R$ is locally invertible, so*

$$\mathrm{SemiReg}(R) = \mathrm{LocInv}(R).$$

A subring R of a (not necessarily unital) ring Q is said to be a *local order* in Q if

$$\mathrm{SemiReg}(R) \subseteq \mathrm{LocInv}(Q) \qquad (\text{hence } \mathrm{SemiReg}(R) = R \cap \mathrm{LocInv}(Q)) \tag{2.4}$$

and

for every $q \in Q$ there exists $x \in$ SemiReg(R) such that $q \in xQx$ and xRx is a classical order in the unital ring $xQx = eQe$ ($e = x^{\#}x$), where $x^{\#}$ denotes the group inverse of x. (2.5)

If R and Q satisfy only (2.5) then we say that R is a *weak local order* in Q.

For a subset S of a ring R we write lann(S) (rann(S)) to denote the *left* (*right*) *annihilator* of S, and ann(S) = lann(S) $\cap$ rann(S). If $S = \{x\}$ consists of a single element, we write lann(x), rann(x), and ann(x).

A right (left) ideal of a ring is *essential* if it has nonzero intersection with every nonzero right (left) ideal of the ring. Put

$$Z_l(R) = \{a \in R\colon \operatorname{lann}(a) \text{ is essential}\}.$$

If $Z_l(R) = 0$ then R is said to be *left nonsingular. Right nonsingular* rings are defined dually. A ring R is *nonsingular* if it is both left and right nonsingular. It is known that $Z_l(R)$ and $Z_r(R)$ are ideals of R. Let us write

$$I_l(R) = \{x \in R\colon Rx \text{ contains no infinite direct sum of nonzero left ideals of } R\}.$$

$I_r(R)$ has a dual meaning. By [12, Proposition 3.5], $I_l(R)$ and $I_r(R)$ are left and right ideals of R respectively. Moreover, for a prime ring R with nonzero socle, $I_l(R) = I_r(R) = \operatorname{Soc}(R)$. In fact, if $a \notin \operatorname{Soc}(R)$ then aR (similarly Ra) contains an infinite sequence of orthogonal division idempotents.

As a consequence of Theorem 1 and Proposition 10 of [2] (see also [12, Theorem 1.1]), the following is obtained:

THEOREM 2. *A ring R is a local order in a simple ring Q satisfying dcc on principal one-sided ideals if and only if R is a prime nonsingular ring satisfying $I_l(R) = R = I_r(R)$.*

Martindale Ring of Quotients. Let R be a semiprime ring. We note that a (two-sided) ideal I of R is essential as a left ideal if and only if it is essential as a right ideal, equivalently, $\operatorname{lann}(I) = \operatorname{rann}(I) = \operatorname{ann}(I) = 0$.

Consider the set of all left R-module mappings $f\colon {}_R I \to {}_R R$ where I ranges over all essential ideals of R. We shall write these maps f on the *right*, and compose them *from left to right*: xf and $x(f \circ g) = (xf)g$. In particular, for us $R_{ab} = R_a \circ R_b$, where $R_a x = xa$. Two such functions are said to be equivalent if they agree on their common domains. Let $[f, I]$ denote the equivalence class of f and let $Q_l(R)$ be the set of all such equivalence classes. This set combined with the addition and the composition defined above is a ring with a unit called the *left Martindale ring of quotients* of R. The mapping $a \to R_a$ is an embedding of R into $Q_l(R)$, and for every $q \in Q_l(R)$ there exists an essential ideal I of R such that $Iq \subseteq R$. Now the *symmetric ring of quotients* is defined as the subring of $Q_l(R)$ given by

$$Q_s(R) = \{q \in Q_l(R)\colon qI + Iq \subseteq R, \text{ for some essential ideal } I \text{ of } R\}.$$

This is the approach followed by Passman in [27], but there is a different approach (see [25]). It can be shown from the definition of $Q_s(R)$ that

$$I_l(Q_s(R)) \cap R \subseteq I_l(R). \tag{2.6}$$

Remark. Let Q be a simple ring with minimal one-side ideals. Note that the socle of $Q_s(Q)$ is precisely Q. In fact, if we represent Q as the simple ring $F_Y(X)$ of finite-rank continuous linear operators relative to a dual pair of vector spaces (X, Y) over a division ring Δ, then $Q_s(Q)$ is the ring $L_Y(X)$ of all continuous linear operators (see [18, 1.15.4] or [5]).

PROPOSITION 3 [9, Proposition 4]. *Let R be a ring which is a local order in a simple ring Q with minimal one-sided ideals. Then the symmetric ring of quotients $Q_s(R)$ of R can be embedded in the symmetric ring of quotients $Q_s(Q)$ of Q. Moreover, $I_r(Q_s(R)) \subseteq I_r(Q_s(Q)) = \mathrm{Soc}(Q_s(Q)) = Q$.*

Let R be a prime ring and set $I(R) = I_l(R) \cap I_r(R)$. The following result shows that if the symmetric ring of quotients of R has nonzero socle, then $I(R)$ is a local order the socle of $Q_s(R)$. More precisely,

THEOREM 4 [9, Theorem 5]. *Let R be a prime ring such that $Q_s(R)$ has nonzero socle. Then $I(R) = R \cap \mathrm{Soc}(Q_s(R))$ (hence an ideal of R) is a local order in the simple ring $\mathrm{Soc}(Q_s(R))$.*

Generalized Identities with Involution. An involution $*: R \to R$ will be called *diagonal* if $aH(R, *)a^* = 0$ implies $a = 0$ $(a \in R)$. Otherwise we say that $*$ is an *alternate* involution. Note that if R is a prime ring with nonzero socle, then every diagonal (respectively, alternate) involution $*$: $R \to R$ can be represented as the adjoint relative to a nondegenerate hermitian and nonalternating (respectively, alternate) product. See the structure theorem for prime rings with involution ([14] or [15]). By using the methods of the proof of [11, Theorem 5.5], the following can be proved:

LEMMA 5. *Let R be a ring with an involution $*: R \to R$ which is a local order in a ring Q. Then there exists a unique involution $*: Q \to Q$ extending that of R. In fact, if $q = a^{\#}b$ then $q^* = b^*(a^*)^{\#}$. Moreover, if $*: R \to R$ is diagonal, then its extension to Q is also diagonal.*

Following [18, p. 81], we denote by $A * B$ the *free product* of (associative) algebras with a unit, A, B, over a field C. Now let R be a prime ring with involution $*$ and let C denote the center of $Q_l(R)$. It is well-known (see [18, 1.5.9]) that C is a field called the *generalized centroid or extended centroid* of R. Consider the set X of formal variables as the disjoint union of two equipotent sets Y and Y^*, where $X_i \to X_i^*$ is a bijection from Y onto Y^*. With this regarding of X, the elements of the C-algebra $Q_l(R) * C\langle X\rangle$ are called *generalized identities with involution*, and R is said to satisfy a given such nontrivial identity p if $f(p) = 0$ for all homomorphisms f: $Q_l(R) * C\langle X\rangle \to Q_l(R)$ of C-algebras such that $f(X) \subseteq R$, $f(q) = q$ for all $q \in Q_l(R)$, and $f(X_i^*) = f(X_i)^*$ for all $X_i \in Y$. For the proof of the following theorem, the reader is referred to [3, pp. 46, 48].

THEOREM 6. *Let R be a prime ring with involution $*$ satisfying a nontrivial generalized identity with involution. Then* (a) *the involution $*$ of R extends to an involution of the symmetric ring of quotients $Q_s(R)$ of R,* (b) *$Q_s(R)$ is a primitive ring with nonzero socle, and* (c) *each generalized identity with involution of R is a generalized identity with involution of $Q_s(R)$.*

We note that every prime ring R with an alternate involution $*: R \to R$ satisfies the generalized identity with involution $p(X, X^*) = a(X + X^*)a^*$, for some nonzero $a \in R$. Since R is prime, $aRa^* \neq 0$ for every nonzero $a \in R$. Hence $p(X, X^*)$ is a nontrivial generalized identity with involution. As a consequence of Theorem 4, the following can be obtained:

THEOREM 7 [9, Theorem 10]. *Let R be a prime ring with an alternate involution $*$. Then $I(R)$ is an ideal of R which is a local order in a simple ring $F_V(V)$, where V is an alternate self-dual vector space over a field K.*

3. ORDERS IN JORDAN ALGEBRAS

From now on all the algebras we consider in this paper are over a ring of scalars ϕ containing $\frac{1}{2}$. Our standard references for Jordan algebras are [16, 17, 32]. Every associative algebra A (with product xy) gives rise to a Jordan algebra A^+ under the new multiplication defined by $x \cdot y = \frac{1}{2}(xy + yx)$. If A has an involution $*$, then the set $H(A, *) = \{a \in A: a = a^*\}$ is a subalgebra of A^+. Let J be a Jordan algebra. For $x, y \in J$ we write their product by $x \cdot y$, and

(i) $U(x) = U_x: J \to J \qquad U_x y = 2x \cdot (x \cdot y) - x^2 \cdot y$

(ii) $\{xyz\} = L(x, y)z = \frac{1}{2}(U_{x+z} - U_x - U_z)y$

(iii) $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$

(iv) $R(x)y = x \cdot y$

for $x, y, z \in J$.

The following identities can be verified by using the Shirshov–Macdonald Theorem [32, p. 78]:

$$U(U_x y) = U_x U_y U_x \tag{3.1}$$

$$x \cdot U_x y = U_x(y \cdot x) \tag{3.2}$$

$$U_x z \cdot (U_x y)^2 = U_x\{z U_{x^2} y y\} \tag{3.3}$$

$$U_x z \cdot (U_x y)^4 = U_x\{z U_{U(x^2)y} y y\} \tag{3.4}$$

By $B \leq J$ (respectively, $B \triangleleft J$) we shall mean that B is a subalgebra (respectively, ideal) of J. An *inner ideal* (*strict inner ideal*) is a subspace I of J such that $U_I J \subseteq I$ ($U_I J' \subseteq I$, where J' denotes the *unital hull* of J). For any subset M of J denote by $K_J(M)$ the inner ideal generated by M. A *direct sum of inner ideals* is a family $\{I_\lambda\}_{\lambda \in \Lambda}$ of nonzero inner ideals such that for each $\mu \in \Lambda$, $I_\mu \cap K_J(\Sigma_{\lambda \neq \mu} I_\lambda) = 0$.

The *annihilator* of a subset M of J is defined to be the set $\mathrm{ann}_J(M)$ of all $z \in J$ satisfying the following equivalent conditions:

(i) $z \cdot M = 0 = (z, J, M)$

(ii) $\{zMJ'\} = 0$,

(iii) $\{MzJ'\} = 0$.

The reader is referred to [28, 6] for general results on annihilators in Jordan algebras. A Jordan algebra is *nondegenerate* if $U_a = 0$ implies $a = 0$. When J is nondegenerate the annihilator of an ideal I of J has an easy expression [30, Theorem 7],

$$\mathrm{ann}_J(I) = \{x \in J \colon U_x I = 0\}. \tag{3.5}$$

Note (see [6]) that for a semiprime associative algebra A, $\mathrm{ann}_{A^+}(M) = \mathrm{lann}(M) \cap \mathrm{rann}(M)$ for any subset M of A. A Jordan algebra J is *prime* if $U_B C = 0$ implies $B = 0$ or $C = 0$, with B, C ideals of J. For a nondegenerate Jordan algebra primeness is inherited by ideals. Moreover, although prime Jordan algebras need not be nondegenerate, this is true for simple Jordan algebras.

Let J be a subalgebra of a Jordan algebra Q. Given $q \in Q$, we say that $x \in J$ is a *J-denominator* for q if

$$x \cdot q + (x, J, q) \subseteq J. \tag{3.6}$$

We denote by $\mathscr{D}_J(q)$ the set of all J-denominator for q. Notice that if $x \in J$ is invertible in Q then x is a J-denominator for x^{-1}. Moreover, $\mathscr{D}_J(q)$ satisfies similar properties to those of the usual annihilator (see [17, p. 8.5]). Thus, we have that

$$\mathscr{D}_J(q) \text{ is a strict inner ideal of } J, \tag{3.7}$$

$$\text{if } x \text{ is a } J\text{-denominator for } q \text{ then } \{qxJ\}, \{xqJ\}, \text{ and } U_x q \text{ all fall in } J, \tag{3.8}$$

$$\text{if } x \cdot q \text{ and } x^2 \cdot q \text{ are in } J, \text{ then } x^2 \text{ is a } J\text{-denominator for } q. \tag{3.9}$$

A subset S of a Jordan algebra J is called a *monad* if $U_s t$ and s^2 are in S, for all $s, t \in S$. We note that our definition of monad is a little stronger than the one given by Zel'manov [30, p. 899], where only the first condition is required. Now J is an order in a Jordan algebra *with a unit* $Q \supseteq J$ *relative to* S (or an *S-order in* Q) if

(i) every element $x \in S$ is invertible in Q,

(ii) each element $q \in Q$ has a J-denominator in S, and

(iii) for all $s, t \in S$, $U_s S \cap U_t S \neq \emptyset$.

We call (iii) the *common multiple property* (CMP) for S. Note that (i) implies $S \subseteq \mathrm{Reg}(J)$, where

$$\mathrm{Reg}(J) = \{x \in J : U_x : J \to J \text{ is injective}\}.$$

A *classical order* is just an S-order for the full monad $S = \mathrm{Reg}(J)$. Note also that if J is unital, it is always an S-order in itself ($Q = J$) relative to $S = \{1\}$.

As a consequence of (3.7) and (3.9) the following can be obtained:

$$\text{If } J \text{ is an } S\text{-order in } Q \text{ and } J \leq K \leq Q \text{ then } K \text{ is an } S\text{-order in } Q. \quad (3.10)$$

PROPOSITION 8. *Let A be an associative algebra which is a classical order in an associative algebra Q with a unit,*

(1) *if $A^+ \leq J \leq Q^+$, then J is an order in Q^+ relative to* $\mathrm{Reg}(A)$; *equivalently, A^+ is a classical order in Q^+.*

Suppose now that A has an involution $*: A \to A$ and let $*$ also denote its unique extension to Q,

(2) *if $H(A, *) \leq J \leq H(Q, *)$, then J is an order in $H(Q, *)$ relative to $H(\mathrm{Reg}(A), *)$.*

Proof. (1) (i) Clearly $\mathrm{Reg}(A) \subseteq \mathrm{Inv}(Q) = \mathrm{Inv}(Q^+)$. (ii) Let $q \in Q$. Since A is a classical order in Q, by (2.3) there exist $a, b, s \in A$, with $s \in \mathrm{Reg}(A)$, such that q is able to write $as^{-1} = s^{-1}b$. By (3.9), s^2 is a J-denominator for q since both $q \cdot s$ and $q \cdot s^2$ are in A. (iii) Suppose now that $s, t \in \mathrm{Reg}(A)$. Write $t^{-1}s = s_2 t_2^{-1}$, $st^{-1} = t_3^{-1}s_3$ for some s_i, t_i in $\mathrm{Reg}(A)$. Then $st_2 = ts_2$, $t_3 s = s_3 t$, and hence $s(t_2 t_3)s = t(s_2 s_3)t \in s\mathrm{Reg}(A)s \cap t\,\mathrm{Reg}(A)t$. (2) (i) is the same as before. (iii) Let $s, t \in H(\mathrm{Reg}(A), *)$. By the above, there exist $a, b \in \mathrm{Reg}(A)$ such that $sas = tbt$. Then $s(as^2a^*)s = t(bt^2b^*)t \in sH(\mathrm{Reg}(A), *)s \cap tH(\mathrm{Reg}(A), *)t$. (ii) Let $q = q^* = as^{-1} = (as^*)(ss^*)^{-1}$ for $ss^* \in H(\mathrm{Reg}(A), *)$. We also have $q = q^* = (ss^*)^{-1}(sa^*)$, so $2q \cdot ss^* = as^* + sa^* \in H(A, *)$, and $2q \cdot (ss^*)^2 = (as^*)(ss^*) + (ss^*)(sa^*) \in H(A, *)$, showing (by 3.9) $(ss^*)^2$ is an $H(A, *)$ (hence J)-denominator for q belonging to $H(\mathrm{Reg}(A), *)$.

The following result has been proved in [31, p. 568]. Nevertheless, for the sake of completeness, we provide here a proof of it.

LEMMA 9. *Let J be a Jordan algebra which is an order in a unital Jordan algebra Q relative to a monad S of J. Then for $x, y \in J \cap \mathrm{Inv}(Q)$, $U_x \operatorname{Reg}(J) \cap U_y \operatorname{Reg}(J) \neq \emptyset$.*

Proof. If x^{-1}, y^{-1} have J-denominators s_1, s_2 in S then, by CMP for S, there is $s \in s_1 S s_1 \cap s_2 S s_2$, and hence s is a J-denominator for x^{-1} and y^{-1} by (3.7). We claim $U_{x^{-1}} s^3 \in J$, since by Macdonald's theorem

$$U_{x^{-1}} s^3 = 4U_{x^{-1},s} s - 2\{U_s x^{-1} s x^{-1}\} - \{U_s x^{-1} x^{-1} s\} \in U_J J$$

$$-\{J s x^{-1}\} - \{J x^{-1} s\} \subseteq J$$

by (3.8). Similarly, $U_{y^{-1}} s^3 \in J$. Hence

$$s^3 = U_x U_{x^{-1}} s^3 = U_y U_{y^{-1}} s^3 \in U_x \operatorname{Reg}(J) \cap U_y \operatorname{Reg}(J).$$

A Jordan algebra Q is said to be *artinian* if it satisfies dcc on all inner ideals. Note [23] that a simple Jordan algebra with *finite capacity* ($1 = e_1 + \cdots + e_n$ is a sum of orthogonal division idempotents) is artinian if and only if it is not the Jordan algebra defined by a quadratic form containing an infinite-dimensional totally isotropic vector space, in short, a *nonartinian quadratic factor*. Conversely, nondegenerate artinial Jordan algebras are unital with finite capacity.

A Jordan algebra J is called a *Goldie algebra* if it satisfies the annihilator chain condition and contains no infinite direct sum of inner ideals. Zel'manov succeeded in giving the following Goldie-type theorem for Jordan algebras.

THEOREM 10 [30, 31]. *A Jordan algebra J is a classical order in a nondegenerate artinian Jordan algebra Q if and only if J is a nondegenerate Goldie Jordan algebra. Moreover, Q is simple if and only if J is prime.*

4. WEAK LOCAL ORDER IN JORDAN ALGEBRAS

Recall that an element x in a Jordan algebra Q is said to be *strongly regular* if $x \in U(x^2)Q$. As it was proved in [21, Lemma 1], $x \in Q$ is strongly regular if and only if it is *locally invertible* ($x \in \mathrm{LocInv}(Q)$) in the sense that there is a (unique) idempotent $e = P(x)$ such that x is invertible (in the usual sense) in the unital Jordan algebra $U_e Q$. Then its inverse

$x^{\#}$ is called the *generalized inverse* of x, and can be characterized (among others) by the following equivalent conditions (see [10]):

(i) $U(x)x^{\#} = x$, $U(x^{\#})x = x^{\#}$, and $U(x^{\#})U(x) = U(x)U(x^{\#})$,

(ii) $U(x)x^{\#} = x$ and $U(x)U(x^{\#})x^{\#} = x^{\#}$,

(iii) $U(x)x^{\#} = x$ and $(x^{\#})^2 \cdot x = x^{\#}$.

We note that the idempotent $e = P(x)$ is given by $U(x)(x^{\#})^2 = U(x^{\#})x^2$.

A subalgebra J of a Jordan algebra Q (not necessarily with a unit) is said to be a *weak local order* in Q if for each $q \in Q$ there exists $x \in \operatorname{LocInv}(Q) \cap J$ such that $q \in U_x Q$ with $U_x J$ being an order in the unital Jordan algebra $U_x Q = U_e Q$ $(e = P(x) \in Q)$ relative to some monad S_x of $U_x J$.

As a consequence of the definition of weak local order and of (3.10), the following can be obtained:

$$\text{If } J \text{ is a weak local order in } Q \text{ and } J \leq K \leq Q \text{ then } K \text{ is a weak local order in } Q. \tag{4.1}$$

LEMMA 11. *Let A be an associative algebra which is a weak local order in an associative algebra Q,*

(1) *if $A^+ \leq J \leq Q^+$, then J is a weak local order in Q^+.*

Suppose now that A has an involution $* : A \to A$ and let $*$ also denote its unique extension to Q,

(2) *if $H(A, *) \leq J \leq H(Q, *)$, then J is a weak local order in $H(Q, *)$.*

Proof. (1) Given $q \in Q$ there exists $x \in A \cap \operatorname{LocInv}(Q)$ such that $q \in U_e Q$ $(e = P(x))$ with xAx being a classical order in the associative algebra eQe. Now we have by Proposition 8(1) that $U_x J$ is an order in $U_e Q^+$ relative to the monad $\operatorname{Reg}(xAx)$, which proves that J is a weak local order in Q^+. The proof of (2) is similar, so we omit it.

PROPOSITION 12. *Let J be a Jordan algebra which is an S-order in a unital Jordan algebra Q. For every $s \in S$, $U_s J$ is an order in Q relative to $U_s S$. In particular, J is a weak local order in Q.*

Proof. (i) $U_s S \subseteq S \subseteq \operatorname{Inv}(Q)$. (iii) Let $s_1, s_2 \in U_s S$. By the CMP for S we have

$$\begin{aligned}\varnothing \neq U_{U(s)s_1} S \cap U_{U(s)s_2} S &= U_s\big(U_{s_1}(U_s S)\big) \cap U_s\big(U_{s_2}(U_s S)\big)\\ &= U_s\Big(\big(U_{s_1}(U_s S)\big) \cap \big(U_{s_2}(U_s S)\big)\Big).\end{aligned}$$

Hence $\varnothing \neq (U_{s_1}(U_s S)) \cap (U_{s_2}(U_s S))$ because U_s is injective. (ii) Given $q \in Q$, set $q' = U_{s^{-1}} q$. Take $r \in S$ a J-denominator for annihilating q'. Since

$U_rS \cap U_{s^2}S \neq \emptyset$ by the CMP for S and $U_rS \subseteq \mathscr{D}_J(q')$ by (3.7), there exists $t \in S$ such that $U_{s^2}t$ is a J-denominator for q'. We will show that $(U_st)^4$ is a U_sJ-denominator for q. By (3.8)

$$q \cdot (U_st)^2 = U_sq' \cdot (U_st)^2 = (\text{by } 3.3)\ U_s\{q'U_{s^2}tt\} \in U_sJ,$$

and

$$q \cdot (U_st)^4 = U_sq' \cdot (U_st)^4 = (\text{by } 3.4)\ U_s\{q'U(U_{s^2}t)tt\} \in U_sJ,$$

because, by (3.7), $U(U_{s^2}t)t$ is again a J-denominator for q'. Hence $(U_st)^4$ is a U_sJ-denominator for q by (3.9). In particular, J is a weak local order in Q: if $q \in Q$ take any $s \in S$, so $s \in S \subseteq J \cap \mathrm{Inv}(Q) \subseteq J \cap \mathrm{LocInv}(Q)$, so by the above $q \in Q = U_sQ$ with U_sJ being an order in Q relative to $U_sS \subseteq U_sJ$.

The following proposition extends to weak local orders, a result stated by Zel'manov [31] for orders.

PROPOSITION 13. *Let J be a Jordan algebra which is a weak local order in a Jordan algebra Q. Then*

(i) *if δ is a derivation of Q such that $\delta(J) = 0$, then $\delta(Q) = 0$*

(ii) $\mathrm{ann}_J(M) = \mathrm{ann}_Q(M) \cap J$ *for every $M \subseteq J$.*

Proof. (i) For $q \in Q$ there exists $x \in J \cap \mathrm{LocInv}(Q)$ such that $q \in U_xQ = U_eQ$ has U_xJ-denominator $s \in S \subseteq U_xJ \cap \mathrm{Inv}(U_eQ)$. Then, by (3.8), $U_sq \in U_xJ \subseteq J$ has

$$0 = \delta(U_sq) = U_s(\delta(q)) + 2\{\delta(s)qs\} = U_s(\delta(q))$$

because $\delta(J) = 0$. But s is invertible in U_eQ, and $\delta(q)$ remains in U_eq (note $q = U_xq'$, so $\delta(q) = \delta(U_xq') = U_x\delta(q') + 2\{\delta(x)q'x\} = U_x\delta(q') \in U_eQ$), so $U_s\delta(q) = 0$ implies $\delta(q) = 0$.

(ii) Clearly $\mathrm{ann}_Q(M) \cap J \subseteq \mathrm{ann}_J(M)$. Conversely, let $z \in \mathrm{ann}_J(M)$ and $m \in M$. Consider the inner derivation $\delta\colon Q \to Q$ defined by $\delta(w) = (z, w, m)$. By the hypothesis $z \in \mathrm{ann}_J(M)$, δ vanishes on J; therefore by (i) δ vanishes on Q, too, so $(z, Q, M) = 0$ and $z \in \mathrm{ann}_Q(M)$.

PROPOSITION 14. *Let J be a nondegenerate Jordan algebra which is a weak local order in a Jordan algebra Q. Then for each $0 \neq q \in Q$ there exists $r \in J \cap \mathrm{LocInv}(Q)$ such that $q \in U_eQ$ $(e = P(r))$ with $0 \neq U_qU_rJ \subseteq J$. In particular,*

(i) *$U_qJ \cap J \neq 0$ for each $q \neq 0$ in Q,*

(ii) *$I \cap J \neq 0$ for each nonzero inner ideal I of Q.*

Proof. Given $0 \neq q \in Q$ there exists $x \in J \cap \mathrm{LocInv}(Q)$ such that $q \in U_x Q$ with $U_x J$ being an order in $U_e Q$ ($e = P(x)$) relative to a monad S of $U_x J$. Let $s \in S$ be a $U_x J$-denominator for q. By (3.7) both s^2 and $U_s U_s q$ are $U_x J$-denominators for q. Now consider the identity (see QJ8″ of [17])

$$U_q U_{s^2} = 2L(q, s^2)L(q, s^2) - L(q, U_{s^2} q).$$

Then by (3.8) $U_q U_{s^2} U_x J \subseteq U_x J$ with $U_q U_{s^2} U_x U_{s^2} J = U_q U_r J$, where $r = U_{s^2} x$ is invertible in $U_e Q$ because so are x and s. Hence $U_q U_r J \neq 0$ since if $U_q U_r J = 0$ then $U(U_r q)J = U_r U_q U_r J = 0$, which implies $U_r q = 0$ because J is nondegenerate which is a contradiction since r is invertible in $U_e Q$.

5. Local Orders in Jordan Algebras with dcc on Principal Inner Ideals

Recall that for a nondegenerate Jordan algebra J the socle ($\mathrm{Soc}(J)$) is defined as the sum of all minimal inner ideals of J [26]. A nondegenerate Jordan algebra J satisfying dcc on principal inner ideals (equivalently, coinciding with its socle [7, 20]) also satisfies acc on $\mathrm{ann}_J(x)$, $x \in J$ [6, 8]. We also note that for a semiprime associative algebra A, $\mathrm{Soc}(A) = \mathrm{Soc}(A^+)$. Moreover, if A has an involution $*: A \to A$, then $\mathrm{Soc}(H(A, *)) = H(\mathrm{Soc}(A), *)$ [4].

Prime nondegenerate Jordan algebras with nonzero socle, and in particular simple Jordan algebras containing minimal inner ideals, were described by Osborn and Racine in [26]. Actually, this result can be derived from Zel'manov's classification of prime nondegenerate Jordan algebras (see [5]). For the sake of completeness and given that simple Jordan algebras with minimal inner ideals play a fundamental role in our work, we include here the statement of the Osborn–Racine Theorem for the case of a simple Jordan algebra.

Theorem 15 [26, 5]. *Let J be a simple Jordan algebra with minimal inner ideals. Then one of the following conditions holds:*

(1) *J is a 27-dimensional exceptional Jordan algebra over its centre,*

(2) *J is the Jordan algebra defined by a nondegenerate quadratic form,*

(3) *$J = F_Y(X)^+$ relative to a dual pair of vector spaces (X, Y) over a division ring,*

(4) *$J = H(F_V(V), *)$, where V is a self-dual vector space (hermitian or alternate) over a division ring with involution, and $*$ is the adjoint involution.*

Remark 16. Note that by the Litoff theorem [1] for Jordan algebras and the structure of inner ideals of Jordan algebras of finite capacity [23], every simple Jordan algebra J with minimal inner ideals and which is not a nonartinian quadratic factor, that is, J is not the Jordan algebra defined by a nondegenerate quadratic form on a vector space containing a totally isotropic infinite dimensional subspace, is *locally artinian* in the sense that for each idempotent e in J, U_eJ is a simple artinian Jordan algebra.

An element x in a Jordan algebra J will be called *semiregular* if $\operatorname{ann}_{J'}(x) = \operatorname{ann}_{J'}(x^2)$ with J' denoting the unital hull of J. By SemiReg(J) we shall mean the set of all the semiregular elements of J. Note that by [6, (5.3)]

$$\text{every element in a Jordan algebra without nonzero nilpotent elements is semiregular.} \tag{5.1}$$

Let J be a subalgebra of a Jordan algebra Q, then

$$\operatorname{LocInv}(J) \subseteq J \cap \operatorname{LocInv}(Q) \subseteq \operatorname{SemiReg}(J) \tag{5.2}$$

since if x is locally invertible in Q and $a \in \operatorname{ann}_{J'}(x^2)$, then (1) $U_x a \in U_{x^2}U_Q(U_{x^2}a) = 0$, (2) $a \cdot x \in a \cdot U_{x^3}Q = 2\{a \cdot x^2 U_x Q x^2\} - U_x\{U_x a Q x^2\} = 0$, and (3) $L(a, x^2) = 2L(a \cdot x, x) - R(U_x a) = 0$. Hence, by (3), $L(a,x) \in L(a, U_{x^2}Q) = -L(Q, U_{x^2}a) + L(\{ax^2Q\}, x^2) = 0$, so $a \in \operatorname{ann}_{J'}(x)$.

The converse is true in nondegenerate Jordan algebras with dcc on principal inner ideals. In fact, we have the following more general result.

PROPOSITION 17. *For an element x in a nondegenerate Jordan algebra J the following conditions are equivalent*:

(i) $x \in \operatorname{LocInv}(J)$,

(ii) *x is semiregular and strongly π-regular, i.e., the descending chain*

$$U(x)J \supseteq U(x^2)J \supseteq \cdots$$

is stationary.

Thus if J has dcc on principal inner ideals, $\operatorname{LocInv}(J) = \operatorname{SemiReg}(J)$.

Proof. (i) $\Rightarrow$ (ii): By the above, every locally invertible element is semiregular. Moreover, $U_xJ = U_{x^2}J$. (ii) $\Rightarrow$ (i): Conversely, let x be semiregular and strongly π-regular and $x = x_1 + x_0$ be the Fitting decomposition of x [21], where x_1 is invertible in U_eJ for a unique idempotent $e \in J$ and $x_0 \in J_0(e)$ is nilpotent. We will show that $x_0 = 0$ so $x = x_1$ is clearly locally invertible. Suppose then that $x_0 \neq 0$. We have two possibilities:

(a) $x_0^2 = 0$. Note that $J_0(e)$ is nondegenerate since J is so. Hence there exists $y_0 \in J_0(e)$ such that $x_0 \cdot y_0 \neq 0$. Then $x \cdot y_0 = (x_1 + x_0) \cdot y_0 = x_0 \cdot y_0 \neq 0$ implies $y_0 \notin \operatorname{ann}_J(x)$, but $x^2 \cdot y_0 = (x_1^2 + x_0^2) \cdot y_0 = x_1^2 \cdot y_0 = 0$, and for all $z \in J$ we have by Peirce relations [16, p. 119] that $(y_0, z, x^2) = (y_0, z, x_1^2) = 0$. Then $y_0 \in \mathrm{ann}_J(x^2)$, which contradicts semiregularity of x.

(b) $x_0^2 \neq 0$. We have $x_0^k = 0$ with $x_0^{k-1} \neq 0$ for some $k \geq 3$ since x_0 is nilpotent. Then $x_0^{k-2} \cdot x = x_0^{k-2} \cdot (x_1 + x_0) = x_0^{k-1} \neq 0$ implies $x_0^{k-2} \notin \mathrm{ann}_J(x)$, but $x_0^{k-2} \cdot x^2 = x_0^k = 0$ and, for all $z \in J$, we have by [16, p. 119] again and [32, p. 68] that $(x_0^{k-2}, z, x^2) = (x_0^{k-2}, z, x_1^2) + (x_0^{k-2}, z, x_0^2) = 0$. Then $x_0^{k-2} \in \mathrm{ann}_J(x^2)$, again contradicting semiregularity of x.

LEMMA 18. *Let Q be a Jordan algebra with a unit. For a strongly π-regular element q in Q the following conditions are equivalent*:

(i) $q \in \mathrm{Inv}(Q)$,

(ii) $q \in \mathrm{Reg}(Q)$,

(iii) $\mathrm{ann}_Q(q) = 0$.

Proof. In general, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Suppose now that q is strongly π-regular. If q is not invertible then q has a Fitting decomposition $q = q_1 + q_2$ relative to an idempotent $e \neq 1$. If $q_0^k = 0 \neq q_0^{k-1}$ then, as in Proposition 17, $q_0^{k-1} \in \mathrm{ann}_Q(q)$. If $q_0 = 0$, $q = q_1$ then $1 - e \in \mathrm{ann}_Q(q)$, which proves that (iii) $\Rightarrow$ (i).

A subalgebra J of a Jordan algebra Q will be said to be a *local order* in Q if

$$\mathrm{SemiReg}(J) \subseteq \mathrm{LocInv}(Q) \text{ and } \tag{5.3}$$

for every $q \in Q$ there exists $x \in \mathrm{SemiReg}(J)$ such that $q \in U_x Q$ with $U_x J$ being a classical order in $U_e Q$ $(e = P(x))$. (5.4)

PROPOSITION 19. *Let J be a nondegenerate Jordan algebra which is a local order in a unital Jordan algebra Q. Then J is a classical order in Q.*

Proof. (i) $\mathrm{Reg}(J) \subseteq \mathrm{Inv}(Q)$: If $x \in \mathrm{Reg}(J)$, then x is semiregular so that by (5.3) x has a generalized inverse $x^\#$ in Q with $e = P(x) = U_x(x^\#)^2$. Write $q = 1 - e$. Then $x \in \mathrm{ann}_Q(q)$ since $x \in Q_1(e)$, $q \in Q_0(e)$ (Peirce decomposition relative to e). Now we claim that $q = 0$, and hence that x is invertible in Q with inverse $x^\#$. If q were different from 0, by Proposition 14 there would be an element $0 \neq t \in U_q J \cap J \subseteq Q_0(e) \cap J$. Then $U_x t \in U(Q_1(e))Q_0(e) = 0$, which is a contradiction because x is regular in J. (ii) Each $q \in Q$ has a regular denominator: Since Q has a unit element, there exists x in $\mathrm{SemiReg}(J)$ such that $1 \in U_x Q$ with $U_x J$ an order in $U_e Q$ relative to $\mathrm{Reg}(U_x J)$. But $1 \in U_x Q$ implies that x is invertible in Q, so $e = P(x) = 1$ and $U_e Q = Q$. Thus $U_x J$ being a classical order in Q implies first that $\mathrm{Reg}(U_x J) \subseteq J \cap \mathrm{Inv}(Q) \subseteq \mathrm{Reg}(J)$, and next that any $q \in Q$ has a $U_x J$-denominator $y \in \mathrm{Reg}(U_x J) \subseteq \mathrm{Reg}(J)$. Hence both $y \cdot q$ and $y^2 \cdot q$ are in $U_x J$. Thus y^2 is a J-denominator for q in $\mathrm{Reg}(J)$ by (3.9). (iii) $\mathrm{Reg}(J)$ has the CMP: this follows by Lemma 9.

An element x in a Jordan algebra J is said to have *finite Goldie dimension*, if $U_x J$ contains no infinite direct sum of inner ideals of J. It follows from Remark 16 that if Q is a simple Jordan algebra with dcc on principal inner ideals and which is not a nonartinian quadratic factor then every element x in Q has finite Goldie dimension. Even in a nonartinian quadratic factor, every noninvertible (i.e., isotropic) element has finite Goldie dimension.

THEOREM 20. *Let J be a Jordan algebra which is a weak local order in a nondegenerate* (*respectively simple*) *Jordan algebra Q with dcc on principal inner ideals. Then*

(i) *J is a local order in Q,*

(ii) *J is a nondegenerate* (*respectively prime nondegenerate*) *Jordan algebra satisfying acc on* $\mathrm{ann}_J(x)$, $x \in J$.

If additionally Q is locally artinian (*equivalently, Q does not contain nonartinian quadratic factors*), *then*

(iii) *every element x in J has finite Goldie dimension.*

Proof. We will give the proof in successive steps. Suppose first that J is a weak local order in a nondegenerate (simple) Jordan algebra Q with dcc on principal inner ideals.

$$J \text{ is a nondegenerate (respectively prime nondegenerate) Jordan algebra.} \tag{5.5}$$

Let $x \in J$ be such that $U_x J = 0$. By the Litoff Theorem for Jordan algebras [1] we can choose an idempotent u in Q such that $x \in U_u Q$ with $U_u Q$ being a nondegenerate (simple if Q is so) [24] Jordan algebra with finite capacity [4, Proposition 4.5]. By definition of weak order, there is $s \in J \cap \mathrm{LocInv}(Q)$ such that $u \in U_e Q$ ($e = P(s)$) with $U_s J$ being an order in the nondegenerate (simple) Jordan algebra of finite capacity $U_e Q$ relative to a monad S of $U_s J$. Now $U_x J = 0$ implies $U(U_s x) U_s J = 0$ so $U_s x = 0$ because by [31, Lemma 2] $U_s J$ is nondegenerate (we remark that only finite capacity is used there). Hence $x = 0$ because s is invertible in $U_e Q$. Similarly it can be proved that if Q is simple then J is a prime Jordan algebra by reducing the problem to the case of an order in a simple Jordan algebra of finite capacity, and then applying [31, Lemma 2] again.

For all $M, N \subseteq J$, $\mathrm{ann}_J(M) \subseteq \mathrm{ann}_J(N)$ if and only if $\mathrm{ann}_Q(M) \subseteq \mathrm{ann}_Q(N)$, and $\mathrm{ann}_J(M) \neq 0$ if and only if $\mathrm{ann}_Q(M) \neq 0$. (5.6)

By Proposition 14, $\mathrm{ann}_Q(M) \subseteq \mathrm{ann}_Q(N)$ implies $\mathrm{ann}_J(M) = J \cap \mathrm{ann}_Q(M) \subseteq J \cap \mathrm{ann}_Q(N) = \mathrm{ann}_J(N)$. Conversely, suppose that $\mathrm{ann}_Q(M)$ is not contained in $\mathrm{ann}_Q(N)$. We will show that $\mathrm{ann}_J(M)$ is not contained in $\mathrm{ann}_J(N)$. Indeed, if $\mathrm{ann}_Q(M)$ is not contained in $\mathrm{ann}_Q(N)$ then $\mathrm{ann}_Q(M)$ contains an element q, generating a minimal inner ideal, which is not in $\mathrm{ann}_Q(N)$. Since annihilators are inner ideals, such an element can be obtained by using the Diagonalization Theorem [22, Proposition 1] and Peirce relations [19, Theorem 5.4.9] for idempotents in Jordan pairs. Since J is nondegenerate by (5.5), $U_q J \cap J \neq 0$ by Proposition 14(i). Take $a \in J$ such that

$$0 \neq U_q a \in J \cap \mathrm{ann}_Q(M) = \mathrm{ann}_J(M).$$

But $U_q a \notin \mathrm{ann}_J(N)$, since otherwise $U_q a \in \mathrm{ann}_J(N) \subseteq \mathrm{ann}_Q(N)$ would imply $q \in U(U_q a)Q \subseteq \mathrm{ann}_Q(N)$ by minimality of the inner ideal $U_q J$, which is a contradiction. Similarly, $\mathrm{ann}_Q(M) = 0$ implies $\mathrm{ann}_J(M) = \mathrm{ann}_Q(M) \cap J = 0$; and conversely, if $\mathrm{ann}_Q(M) \neq 0$ then $\mathrm{ann}_J(M) \neq 0$ by above.

$$\mathrm{SemiReg}(J) \subseteq \mathrm{LocInv}(Q). \tag{5.7}$$

Let $x \in J$ be such that $\mathrm{ann}_{J'}(x) = \mathrm{ann}_{J'}(x^2)$. Then $\mathrm{ann}_{Q'}(x) = \mathrm{ann}_{Q'}(x^2)$ by (5.6). Hence x is locally invertible in Q by Proposition 17. By [6, 8], Q satisfies acc on annihilators of single elements. Hence we obtain from (5.6) again

$$J \text{ satisfies acc on } \mathrm{ann}_J(x),\ x \in J. \tag{5.8}$$

LEMMA 21. *Suppose now that T is a Jordan algebra which is an order in a nondegenerate Jordan algebra K of finite capacity, relative to a monad S of T. Then T is a classical order in K.*

Proof. By Lemma 9, we need only show that every regular element in T is invertible in K. Let $x \in \mathrm{Reg}(T)$. Then $\mathrm{ann}_T(x) = 0$ and hence by (5.6) (note that this is applicable since by Proposition 12 T is a weak local order in K) $\mathrm{ann}_K(x) = 0$, which implies by Lemma 18 (recalling that all elements of K are strongly π-regular by the dcc on principal inner ideals) that x is invertible in K.

Return to the proof of Theorem 20. It follows from (5.7) and Lemma 21 that

$$J \text{ is a local order in } Q. \tag{5.9}$$

Suppose additionally that Q is locally artinian. Then

$$\text{Every element } x \text{ in } J \text{ has finite Goldie dimension.} \tag{5.10}$$

Given $x \in J$, by definition of weak local order, there exists $s \in J \cap \operatorname{LocInv}(Q)$ such that $x \in U_s Q$ with $U_s J$ being an order in the nondegenerate *artinian* Jordan algebra $U_e Q$ $(e = P(s))$ because Q is locally artinian. If $U_x J \subseteq J \cap U_e Q$ contains an infinite direct sum $\{I_\alpha\}$ of inner ideals of J, then we have by Proposition 14(ii) [applied with J replaced by $U_s J$ (which is a weak local order by Proposition 12), Q replaced by $U_e Q$, and $0 \neq q_\alpha \in I_\alpha$ so $0 \neq U(q_\alpha) U_s J \cap U_s J \subseteq I_\alpha \cap U_s J$] that $0 \neq I_\alpha \cap U_s J$ form an infinite direct sum of inner ideas of $U_s J$, which is a contradiction because $U_s J$ contains no infinite direct sum of inner ideals [31], which completes the proof of Theorem 20.

Recall that an ideal I of a Jordan algebra J is called *essential* if $I \cap B \neq 0$ for any nonzero ideal B of J, equivalently $\operatorname{ann}_J(I) = 0$. Clearly, nonzero ideas of prime Jordan algebras are essential, and by (3.5) an ideal I of a nondegenerate Jordan algebra J is essential if and only if $U_a I = 0$ implies $a = 0$, $a \in J$.

PROPOSITION 22. *Let J be a Jordan algebra which is a classical order in a nondegenerate Jordan algebra Q of finite capacity n. If I is an essential ideal of J, then I is also a classical order in Q.*

Proof. (ii) Given $q \in Q$ take $s \in \operatorname{Reg}(J)$ a J-denominator for q. By the Diagonalization Theorem [22, Proposition 1] (applied to $s \in Q$), $s = v_1 + \cdots + v_n$ where $e_i = (v_i, w_i)$ are orthogonal division idempotents of the Jordan pair (Q, Q); but J is a weak local order in Q (Proposition 12), so by Proposition 14(i) $U_{w_i} J \cap J \neq 0$ for each w_i. Take $0 \neq x_i \in U_{w_i} J \cap J$. Then $U_{x_i} I \neq 0$ (by (3.5)) because I is essential. Hence there exists $0 \neq y_i \in I \cap U_{w_i} J$ for each w_i. Now we have by Peirce relations [19, Theorem 5.4.9] for idempotents in Jordan pairs that

$$t := U_{v_1} y_1 + \cdots + U_{v_n} y_n = U_s(y_1 + \cdots + y_n) \in I.$$

Then t is a J-denominator for q by (3.7), which is regular in J because it has rank equal to n in Q and hence, by [22, Corollary 1], it is invertible in Q. Consider now the derivation $\delta\colon J \to J$ given by $\delta(x) = (q, x, t)$. Then, for $m \geq 2$,

$$\delta(t^m) = (q, t^{m-1} \cdot t, t) = t^{m-1} \cdot \delta(t) + t \cdot \delta(t^{m-1}) \in I.$$

Hence $q \cdot t^3 = (q \cdot t^2) \cdot t - \delta(t^2) \in I$ and similarly $q \cdot t^6 \in I$, which implies by (3.9) that t^6 is an I-denominator for q with $t^6 \in I \cap \mathrm{Reg}(J)$. Set $S = I \cap \mathrm{Reg}(J)$. (i) $S \subseteq \mathrm{Reg}(J) \subseteq \mathrm{Inv}(Q)$. (iii) Let $t, s \in S$. By CMP applied to t^2, s^2 in $\mathrm{Reg}(J)$, there exist $x, y \in \mathrm{Reg}(J)$ such that $U_{t^2}x = U_{s^2}y$. Hence $U_t x' = U_s y'$ with $x' = U_t x$, $y' = U_s y$ belong to S. Altogether we have proved that I is an S-order, and hence a classical order by Lemma 21, in Q.

THEOREM 23. *Let J be a Jordan algebra which is a local order in a simple Jordan algebra Q satisfying dcc on principal inner ideals. If I is a nonzero ideal of J then I is also a local order in Q.*

Proof. Given $q \in Q$ there exists $s \in \mathrm{SemiReg}(J)$ such that $q \in U_s Q = U_e Q$ $(e = P(s))$ with $U_s J$ being a classical order in $U_e Q$. Set $T = U_s J$ and $K = U_e Q$. Then T is a classical order in the simple (see [24]) Jordan algebra K of finite capacity (see [4, Proposition 4.5]), with $B = U_s I$ a nonzero (by (3.5) and primeness of J) ideal of T (and hence essential). Then, by Proposition 22, B is a classical order in K. Now by Proposition 14, for every $t \in \mathrm{Reg}(B)$, $U_t B$ is an order, and hence (by Lemma 21 again) a classical order in K. Therefore we have proved that given $q \in U_e Q = K$ there exists $t \in I \cap \mathrm{Inv}(U_e Q) \subseteq I \cap \mathrm{LocInv}(Q)$ such that $U_t U_s I$ is a classical order in $U_e Q$ $(e = P(s) = P(t))$. Now

$$U_t U_s I \leq U_t I \leq U_e Q$$

implies by (3.10) and Lemma 21 that $U_t I$ is a classical order in $U_e Q$, which proves that I is a weak local order, and hence (by Theorem 20(i)), a local order in Q.

6. JORDAN ALGEBRAS WITH LOCAL GOLDIE CONDITIONS

Let J be a Jordan algebra which is a weak local order in a simple Jordan algebra Q with dcc on principal inner ideals and which is not a nonartinian quadratic factor. Then, by Theorem 20, J is a prime nondegenerate Jordan algebra satisfying:

(i) acc on $\mathrm{ann}_J(x)$, $x \in J$,

(ii) for each x in J, $U_x J$ contains no infinite direct sum of inner ideals of J.

The aim of this section is to prove the converse of this result. In fact we prove a slightly more general one.

For any element u in a Jordan algebra J, since annihilators are inner ideals, $\operatorname{ann}_J(u) \subseteq \operatorname{ann}_J(x)$ for $x \in U_u J$. Now a nonzero element $u \in J$ is called *uniform* if

$$\operatorname{ann}_J(u) = \operatorname{ann}_J(x) \qquad \text{for any } 0 \neq x \in U_u J.$$

Note that if $u \in J$ is uniform then every nonzero element in the inner ideal generated by u is uniform as well. Clearly every nonzero element $u \in J$ such that $\operatorname{ann}_J(u)$ is maximal is a uniform element. In particular, if J satisfies acc on the annihilators of its elements, then every nonzero inner ideal of J contains a uniform element. Moreover, by using socle theory (see [20, 22]) it is not difficult to see that an element x in the socle of J is uniform if and only if x is minimal; that is, it generates a minimal inner ideals.

LEMMA 24. *Suppose now that J is a nondegenerate Jordan algebra containing a uniform element u, and let I be an essential ideal of J. Then I has a uniform element.*

Proof. By (3.5) $U_u I \neq 0$ so I contains a uniform element v of J by above. But v is actually uniform in I. Indeed, by [30, Lemma 9], for $x \neq 0$ in $U_v I$,

$$\operatorname{ann}_I(x) = \operatorname{ann}_J(x) \cap I = \operatorname{ann}_J(v) \cap I = \operatorname{ann}_I(v).$$

THEOREM 25. *Every prime nondegenerate Jordan algebra J containing a uniform element and such that every element x in J has finite Goldie dimension is a local order in a simple Jordan algebra Q with dcc on principal inner ideals and which is not a nonartinian quadratic factor.*

Proof. The proof of the theorem will be divided into four steps.

Step 1. By Zelmanov's structure theorem for prime nondegenerate Jordan algebras [29], J is one of the following:

(1) the center Z of J is nonzero and the central localization $Z^{-1}J$ of J is a simple 27-dimensional exceptional Jordan algebra over its center,

(2) $Z \neq 0$ and $Z^{-1}J$ is the simple Jordan algebra defined by a nondegenerate quadratic form,

(3) J contains an ideal I isomorphic to the Jordan algebra A^+, where A is a prime associative algebra, such that

$$A^+ \triangleleft J \leq Q_s(A)^+,$$

(4) J contains an ideal I isomorphic to the Jordan algebra $H(A, *)$,

where $(A, *)$ is a prime associative algebra with involution which is a $*$-envelope of $H(A, *)$ such that

$$H(A, *) \triangleleft J \leq H(Q_s(A), *).$$

In both cases (1) and (2) J is an order relative to $S = Z$, and hence a local order by Proposition 12 and Theorem 20(i), in the simple Jordan algebra $Q = Z^{-1}J$ with dcc on principal inner ideals. In case (1), since inner ideal are invariant under the center [23, Proposition 2] (by von Neumann regularity of Q), we have $Q = Z^{-1}J$ is a simple artinian Jordan algebra. In case (2), Q is also artinian, since otherwise it would contain an infinite-dimensional totally isotropic subspace [23, Corollary of Theorem 6] and hence an infinite direct sum of inner ideals. Take a nonzero element z in the center of J and consider the inner ideal U_zJ which is an order in Q, and a weak local order by Proposition 12 relative to U_zZ. Then, by Proposition 14(ii), U_zJ would contain an infinite direct sum of inner ideals of J, which contradicts the hypothesis that z has finite Goldie dimension.

Step 2. Suppose then that J is as in case (3)

$$A^+ \triangleleft J \leq Q_s(A)^+,$$

for a prime associative algebra A.

If every element in J has finite Goldie dimension then (i) $J \subseteq I(Q_s(A))$ and (ii) $A = I(A)$. (6.1)

(i) Let $x \in J$ and suppose that $Q_s(A)$ has an infinite direct sum $\oplus \rho_i$ of nonzero right ideals contained in $xQ_s(A)$. We claim that the $K_i := \rho_i \cap U_xJ$ form an infinite direct sum of nonzero inner ideals of J, which is a contradiction. Clearly K_i is an inner ideal of J, and since $K_i \subseteq \rho_i \cap J$, they form a direct sum, so we need only to see that every K_i is nonzero. Take $0 \neq xp_i \in \rho_i$, with $p_i \in Q_s(A)$. Then there exists a nonzero ideal M_i of A such that both $0 \neq xp_iM_i$ and p_iM_i are contained in A. Now $0 \neq xp_iM_iAx \subseteq \rho_i \cap U_xA \subseteq \rho_i \cap U_xJ = K_i$, since A is prime. Therefore x lies in $I_r(Q_s(A))$. Similarly, $x \in I_l(Q_s(A))$. Thus $J \subseteq I(Q_s(A))$.

(ii) By (i), $A \subseteq I(Q_s(A)) \cap A \subseteq I(A)$ by (2.6).

Zel'manov has proved [30, p. 903] that if J is a prime nondegenerate Jordan algebra as in (3) satisfying chain conditions on annihilators (acc and dcc are equivalent on annihilators) then A is left or right nonsingular. We improve this result.

Let J be a prime nondegenerate Jordan algebra as in (3). If J contains a uniform element, then A is nonsingular. (6.2)

By Lemma 24, A^+ contains a uniform element. Now we have

PROPOSITION 26. *Let A be a prime associative algebra such that A^+ contains a uniform element. Then A is nonsingular.*

Proof. We will prove that A is left nonsingular; then right nonsingularity of A follows by symmetry. Suppose that $Z_l(A) \neq 0$. Then by Lemma 24, $Z_l(A)$ contains a uniform element u. By semiprimeness of A, there exists $a \in A$ such that $uau \neq 0$; but au lies in $Z_l(A)$ because $Z_l(A)$ is an ideal, so $\operatorname{lann}(au) \cap Au \neq 0$. Hence there is y in A such that $yu \neq 0$ and $yuau = 0$. Again, by semiprimeness of A, $yuzyu \neq 0$ for some $z \in A$. We have proved the existence of an element $c = zy$ such that

$$ucu \neq 0 \qquad \text{and} \qquad \operatorname{rann}(ucu) \neq 0.$$

As above, $0 \neq cu \in Z_l(A)$ implies that there exists t in A such that $tu \neq 0$ but $tucu = 0$. Let $0 \neq r \in \operatorname{rann}(ucu)$. By primeness of A, $rstu \neq 0$ for some $s \in A$. Hence rst does not belong to $\operatorname{ann}_{A^+}(u)$, but $rst \in \operatorname{ann}_{A^+}(ucu)$, which contradicts that u is uniform.

Return to the proof of Theorem 25. If J is as in case (3) then it follows from (6.1)(ii), (6.2), and Theorem 2 that A is a local order in some simple associative algebra Q with minimal one-sided ideals. Hence by (6.1)(i) and Proposition 3, $A^+ \triangleleft J \leq Q^+$. Then by Lemma 11(1) J is a weak local order in Q^+, and hence by Theorem 20(i) J is a local order in Q^+.

Step 3. Suppose now that J is as in diagonal case (4)

$$H(A, *) \triangleleft J \leq H(Q_s(A), *)$$

where $(A, *)$ is a prime associative algebra with diagonal involution.

If J contains a uniform element then A is nonsingular. (6.3)

Suppose otherwise that $Z_r(A) \neq 0$ and let x be a uniform element of J. Let $a \in H(Z_r(A), *)$. Then $xA \cap \operatorname{rann}(a) \neq 0$ since $\operatorname{rann}(a)$ is essential by definition of $a \in Z_r(A)$, and hence $axy = 0$ with $xy \neq 0$ for some $y \in A$, which can even be chosen to be symmetric. Indeed, since the involution is diagonal, $xyhy^*x \neq 0$ for some $h \in H(A, *)$. Then $z = yhy^*$ is the required element. Clearly $z \notin \operatorname{ann}_J(x)$ but $z \in \operatorname{ann}_J(U_x a)$, so by uniformity of x we must have $U_x a = 0$ for every $a \in H(Z_r(A), *)$. But then $xZ_r(A) = 0$ (if $z \in Z_r(A)$ then $(xz)H(A, *)(xz)^* \subseteq U_x H(Z_r(A), *) = 0$, so by diagonality $xz = 0$), which contradicts primeness of A.

(i) If every element in J has finite Goldie dimension, then $J \subseteq I(Q_s(A))$. (ii) If A is a $*$-envelope of $H(A, *)$ and every element in J has finite Goldie dimension, then $A = I(A)$. (6.4)

(i) As in the proof of (6.1)(i), let $x \in J$ and suppose that $Q_s(A)$ has an infinite direct sum $\oplus \rho_i$ of nonzero right ideals contained in $xQ_s(A)$. We claim that $K_i := \rho_i \cap U_x J$ form an infinite direct sum of nonzero inner ideals of J, which will be a contradiction. Clearly K_i is an inner ideal of J, and since $K_i \subseteq \rho_i \cap J$ they form a direct sum, so we need only to see that each K_i is nonzero; but this follows as in (6.1)(i). Take $y_i \in M_i$ such that $0 \neq xp_i y_i$ and $p_i y_i \in A$. Since $*$ is diagonal, $0 \neq xp_i y_i H(A, *)(p_i y_i)^* x \subseteq \rho_i \cap U_x H(A, *) \subseteq \rho_i \cap U_x J = K_i$. Thus $x \in I_r(Q_s(A))$. Similarly, it is proved that $x \in I_l(Q_s(A))$.

(ii) By (i), $J \subseteq I(Q_s(A))$ and hence $H(A, *) \lhd J \subseteq I(Q_s(A)) \cap A \subseteq I(A)$ by (2.6), which implies that $A = I(A)$ because A is generated by $H(A, *)$.

It follows from (6.3), (6.4)(ii), and Theorem 2 that A is a local order in some simple associative algebra Q with minimal one-sided ideals. Moreover, by Lemma 5 the involution $*: A \to A$ has a unique extension $*: Q \to Q$ which is also diagonal. Now we have by (6.4) and Proposition 3 that $H(A, *) \lhd J \leq H(Q, *)$. Then, by Lemma 11(2), J is a weak local order, and hence a local order in $H(Q, *)$ by Theorem 20(i). Note that by the structure theorem for prime rings with involution containing minimal one-sided ideals (see [14] or [15]), Q is the simple ring $F_V(V)$ of all finite rank continuous linear operators relative to a hermitian self-dual vector space V over a division ring with involution $(\Delta, -)$, and $*: Q \to Q$ is the adjoint involution.

Step 4. Suppose finally that J is a prime nondegenerate Jordan algebra as in alternate case (4),

$$H(A, *) \lhd J \leq H(Q_s(A), *),$$

where $(A, *)$ is a prime associative algebra with an alternate involution which is a $*$-envelope of $H(A, *)$. Then A satisfies the generalized identity with involution $p(X, X^*) = a(X + X^*)a^*$, for some nonzero $a \in A$. Hence by Theorem 6, $Q_s(A)$ has nonzero socle and satisfies the same generalized identity with involution. Now, by the structure theorem for prime rings with involution containing minimal one-sided ideals (see again [14] or [15])

(i) $(F_V(V), *) \lhd (Q_s(A), *) \leq (L_V(V), *)$

(ii) $\mathrm{Soc}(Q_s(A)) = F_V(V)$

where $(V, \langle .,. \rangle)$ is an alternate self-dual vector space over a field F, with $*$ being the adjoint involution.

If each element x in J has finite Goldie dimension, then $J \leq H(F_V(V), *)$. (6.5)

Let $q \in J$. If q does not have finite rank then $\mathrm{Im}(q)$ contains an infinite sequence of pairs $\{x_n q, y_n q\}$ of linearly independent vectors, $x_n, y_n \in V$. Given $x, y \in V$, we denote by $x \otimes y$ the rank one continuous linear operator defined by $x'(x \otimes y) = \langle x', x\rangle y$, $x' \in V$. Then $(x \otimes y)^* = -(y \otimes x)$ with $a(x \otimes y) = (xa^* \otimes y)$, $(x \otimes y)a = (x \otimes ya)$ for every continuous linear operator $a \in L_V(V)$. For each positive integer n, set $p_n = x_n \otimes y_n - y_n \otimes x_n \in H(F_V(V), *)$, and write $r_n := U(q)p_n = x_n q \otimes y_n q - y_n q \otimes x_n q$. We note that each r_n is nonzero since $x_n q$, $y_n q$ are linearly independent. In fact, $\mathrm{Im}(r_n) = Fx_n q + Fy_n q$, i.e., $\mathrm{rank}(r_n) = 2$. Moreover,

$$\text{(I)} \qquad U_q U_{p_n} H(Q_s(A), *) = Fr_n.$$

A contradiction will be obtained by showing that $K_n := Fr_n \cap U_q J$ form an infinite direct sum of inner ideals of J. By Theorem 7 and Lemma 11(2), $H(I(A), *)$ is a weak local order in $H(Q, *)$ where we are writing $Q = F_V(V)$. Hence, by Proposition 14, there exists $s_n \in H(I(A), *) \cap \mathrm{LocInv}(H(Q, *))$ such that $p_n \in U_{e_n} H(Q, *)$ with $e_n = P(s_n)$ and

$$\text{(II)} \qquad 0 \neq U_{p_n} U_{s_n} H(I(A), *) \subseteq H(I(A), *).$$

Now $r_n = qp_n q = qp_n(s_n)^2(s_n^\#)^2 q$ implies $\mathrm{rank}(qp_n(s_n)^2) > 1$ since $\mathrm{rank}(r_n) = 2$. Hence, by [23, Lemma 6], $qp_n(s_n)^2 H(Q, *)(s_n)^2 p_n q \neq 0$. But $Q = F_V(V) \triangleleft (Q_s(A), *)$, so by Theorem 6(c) A does not satisfy the generalized identity with involution

$$qp_n(s_n)^2(X + X^*)(s_n)^2 p_n q.$$

Hence,

$$0 \neq qp_n(s_n)^2 H(A, *)(s_n)^2 p_n q \subseteq qp_n s_n H(I(A), *) s_n p_n q$$

(since $s_n \in H(I(A), *)$ and $I(A)$ is an ideal of A by Theorem 7; note that we used the square $(s_n)^2$ so that we would still have an s_n left at this point)

$$\subseteq U_q H(I(A), *) \cap U_q U_{p_n} H(Q_s(A), *) \qquad \text{(by (II))}$$

$$\subseteq U_q J \cap Fr_n = K_n \qquad \text{(by (I))}.$$

Now for each n the inner ideal of $H(Q_s(A), *)$ generated by $\sum_{m \neq n} Fr_m$ is

$$K_{H(Q_s(A), *)}\Big(\sum_{m \neq n} Fr_m\Big) = K_{H(F_V(V), *)}\Big(\sum_{m \neq n} Fr_m\Big) = H(Y \otimes Y, *),$$

where $Y = \sum_{m \neq n} Fx_n q + Fy_n q$ is the subspace of V generated by the images of all r_m $(m \neq n)$ and $H(Y \otimes Y, *)$ the linear span of all the operators $y \otimes y' - y' \otimes y, y, y' \in Y$. Clearly $H(Y \otimes Y, *) \cap Fr_n = 0$ because $\mathrm{Im}(r_n) \cap Y = 0$, which proves that $\{Fr_n\}$ form a direct sum of inner ideals of $H(Q_s(A), *)$. Then $\{K_n\}$ form an infinite direct sum of inner ideals of J since $K_n \subseteq Fr_n$, which contradicts that q has finite Goldie dimension.

$$A \text{ is a local order in } F_V(V). \tag{6.6}$$

By (6.5), $H(A, *) \triangleleft J \leq H(F_V(V), *)$, which implies that $A \subseteq F_V(V)$ because A is generated by $H(A, *)$. We noted $\mathrm{Soc}(Q_s(A)) = F_V(V)$ in (ii) (before (6.5)), so by Theorem 4, $I(A) = A \cap F_V(V) = A$ is a local order in $F_V(V)$.

Then it follows from (6.5), (6.6), and Lemma 11(2) that J is a weak local order, and hence a local order by Theorem 20(i), in the simple Jordan algebra $H(F_V(V), *)$, which completes the proof.

Remark. We end by showing how the Zel'manov theorem for prime Goldie Jordan algebras can be derived from our results. In fact, we prove that every prime nondegenerate Jordan algebra J containing a uniform element and no infinite direct sum of inner ideals is a classical order in a simple artinian Jordan algebra. Indeed, by Theorem 25 J is a local order in a simple Jordan algebra Q with dcc on principal inner ideals; but by Proposition 14(ii) Q cannot contain infinite direct sums of inner ideals, so (see Theorem 15) Q is artinian, and in particular a unital Jordan algebra. Finally, J is a classical order in Q by Proposition 19.

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